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Twisted local systems solve the (holographic) loop equation of large-N QCD_4

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Abstract: We construct a holographic map from the loop equation of large-N QCD in $d = 2$ and $d = 4$, for planar self-avoiding loops, to the critical equation of an equivalent effective action. The holographic map is based on two ingredients: an already proposed holographic form of the loop equation, such that the quantum contribution is reduced to the evaluation of a regularized residue; a new conformal map from the region encircled by the based loop to a cuspidal fundamental domain in the upper half-plane, such that the regularized residue vanishes at the cusp which is the image of the base point of the loop. The critical equation of the holographic effective action determines a unitary abelian local system in $d = 2$ and a non-abelian twisted local system in $d = 4$. As a check in the $d = 2$ theory, we study the distribution of eigenvalues of the Wilson loop implied by the critical equation. As a check in the $d = 4$ theory, we study the first coefficient of the beta function implied by the holographic loop equation and, as a preliminary step, that part of the second coefficient which arises from the rescaling anomaly, in passing from the Wilsonian to the canonically normalised (holographic) effective action.

Keywords: Renormalization Group, 1/N Expansion, Differential and Algebraic Geometry, Non-Commutative Geometry.

Contents

1. Introduction

This paper grew out of the attempt to obtain directly from the loop equation [1, 2] for self-avoiding loops the celebrated formula for the distribution of the eigenvalues of a Wilson loop of area A in the weak coupling phase, in the large-N limit of two dimensional QCD on a sphere, for $g^2 A$ small with respect to $g^2 A_{\text{sphere}}$:

$$W(A) = \int \prod d\theta_i \prod_{i \neq j} |\theta_i - \theta_j| \exp\left(- \frac{N}{2g^2 A} \sum_i \theta_i^2 \right) \sum_i N^{-1} \cos\theta_i \, . \tag{1.1}$$

An essentially equivalent formula was first obtained from the Wilsonian lattice action using functional techniques [3] and afterward eq. (1.1) was derived from the "heat kernel action" [4, 5] or from the semi-circle law for the eigenvalues of free random variables [6], which are employed in the complete operatorial solution of the loop equation in $d = 2$ for arbitrary self-intersections [7]. We present here a derivation based on the loop equation restricted to self-avoiding loops but that extends to the (planar) four dimensional case. Restricting to (planar) self-avoiding loops has some virtue, since the general solution for arbitrary self-intersection seems to be out of the reach of our methods in $d = 4$ and in $d = 2$ as well. In this respect, already long ago, the loop equation in $d = 2$ and in

$d = 4$ has been written in terms of the distribution of the eigenvalues for self-avoiding loops [8].[1] In eq. (1.1) and in this paper we have ignored the contribution of the exterior of the loop, assuming that the internal region is much smaller than its complement. In addition we have not taken into account the periodicity of the eigenvalues. We believe that the techniques of this paper can be extended to include also the more general case. However for simplicity we will consider it elsewhere. In our derivation there are two basic ingredients. The first one is a holographic form of the loop equation for planar loops in $d = 2$ and $d = 4$. This holographic loop equation was derived in [9]. Using by analogy the language of the correspondence between the boundary $\mathcal{N} = 4$ four dimensional gauge theory and the bulk five dimensional super-gravity (string theory) [10], by holography in this context we mean a correspondence between the loop equation, that we think as a theory defined on boundary curves, and an equivalent effective action for the eigenvalues of a Wilson loop, that we think as a theory defined on the bulk. The holographic form of the loop equation that we refer to is a preferred form of the loop equation obtained by means of appropriate changes of variable so that this boundary-bulk correspondence becomes almost manifest, for the reason that the quantum term in the loop equation, given as usual by a contour integral along the loop, is computed as a regularized residue, loop independent for self-avoiding loops. This loop independence makes the holographic loop equation close to admit an equivalent (holographic) effective action. To this form of the loop equation we associate a "classical" holographic effective action, Γ. Γ is "classical" in the sense that the loop equation for Γ still contains a non-vanishing quantum contribution given by the regularized residue. Yet, Γ already contains quantum corrections and indeed it is related to the quantum holographic effective action, Γ_q, by means of a suitable conformal mapping and gauge fixing. In fact the second ingredient needed to complete the holographic correspondence is a conformal map of the region encircled by the based loop that occurs in the loop equation to a cuspidal fundamental domain. On such domain an effective action for the distribution of the eigenvalues of the Wilson loop, Γ_q, can be indeed constructed, because the regularized residue vanishes. Hence the holographic loop equation becomes equivalent to a critical equation for Γ_q. The second ingredient, i.e. the conformal map, is essentially new in this paper with respect to [9] although it already appeared in implicit form in appendix C in [9]. Yet, there, the two-steps logic of holography and conformally mapping was somehow mixed so that we did not realize that the two dimensional technique of appendix C could be in fact extended to the four dimensional case as well. More precisely, our quantum holographic effective action furnishes a critical equation for the eigenvalues of the curvature of a unitary abelian local system [11 − 15] in the two dimensional case and for the eigenvalues of the curvature of a non-unitary non-abelian twisted local system in the four dimensional case. By a twisted local system we mean here a central extension of a (possibly infinite dimensional) representation of the fundamental group of a punctured Riemann surface. Thus, while the curvature of the local system determines directly the eigenvalues of the Wilson loop in $d = 2$, it does it only indirectly, via the non-abelian gauge connection, in $d = 4$.

[1] We thank the referee for pointing out ref. [8] to our attention.

To summarise, the basic ingredients of our holographic correspondence are the following ones.

A holographic form of the loop equation in $d = 2$ and $d = 4$ as described in [9]. This form of the loop equation follows from changes of variable, that though have a geometric meaning in terms of symplectic reduction to a microcanonical ensemble, followed by a choice of a holomorphic gauge, do not involve in a crucial way algebro-geometric concepts. These are rather needed in the assignment of a local system with a lattice of punctures in the $d = 2$ theory and of an infinite dimensional twisted local system with a lattice of punctures in the $d = 4$ theory, so that in both cases at least one of the punctures of such systems lives at the distinguished point of the based loop that enters the loop equation.

A conformal map of the region inside the based loop to a cuspidal fundamental domain.

A regularization of the loop equation at the cusps, that amounts in fact to a compactification at infinity of the cusps.

A reduction of the curvature of the local system at the punctures to an abelian sub-algebra in $d = 2$ and to a Borel sub-algebra in $d = 4$, that follows by mapping conformally all the cusps (but one) along the same line on the boundary of the upper-half plane, by the choice of an axial gauge in a direction orthogonal to the boundary of the upper-half plane and by the compactification of the cusps at infinity.

It may be interesting to observe that the construction of the critical equation for the quantum holographic effective action, which implies the holographic loop equation, involves considering loops with a marked point taken away and compactified at infinity after a change of the conformal structure. From this point of view, the critical equation is the boundary theory defined on the zero dimensional boundary of the loop (i.e. the marked point), while the loop equation defines the theory on the bulk. The definition and the meaning of all these ingredients will be explained in the following sections. In section 2 we recall the holographic form of the loop equation following [9]. This section is needed to make this paper self-contained. To check this form of the loop equation we reproduce from it to lowest order in perturbation theory the propagator in the $d = 2$ case. In section 3 we apply our algebro-geometric techniques to the two dimensional case, in order to derive from the loop equation for self-avoiding loops the distribution of the eigenvalues that has been mentioned as the motivation of this paper. In section 4 we adapt the techniques of section 3 to the four dimensional case and we compute the classical and quantum holographic effective action in terms of functional determinants. In section 5 we compute β_0, the first coefficient of the beta function, from the four dimensional classical holographic effective action, finding exact agreement with the perturbative result. In section 6 we compute the contribution to β_1, the second coefficient of the beta function, that arises from the rescaling anomaly which occurs in passing from the Wilsonian to the canonically normalised form of the holographic effective action. This computation has some interest in itself and it is largely independent of the entire holographic construction. It is also perhaps related to a question about the beta function of QCD raised in [16]. In section 7 we collect some miscellaneous observations, referring especially to analogies in the existing literature. In section 8 we state our conclusions.

2. The holographic loop equation

We can summarise the basic philosophy in [9] as follows. The loop equation in its usual form follows from the observation that the integral of a derivative vanishes:

$$\begin{aligned} 0 &= \int DA_\mu \operatorname{Tr} \frac{\delta}{\delta A_\nu(z)} \left(\exp\left(-\frac{N}{4g^2} \int \operatorname{Tr} F_{\mu\nu}^2 d^4x \right) \Psi(x,x;A) \right) \\ &= \int DA_\mu \exp\left(-\frac{N}{4g^2} \int \operatorname{Tr} F_{\mu\nu}^2 d^4x \right) \times \\ &\quad \times \Bigg(\operatorname{Tr}\left(\frac{N}{g^2} D_\mu F_{\mu\nu}(z) \Psi(x,x;A) \right) + \\ &\qquad + i \int_{C(x,x)} dy_\nu \delta^{(d)}(z-y) \operatorname{Tr}(\lambda^a \Psi(x,y;A) \lambda^a \Psi(y,x;A)) \Bigg) \end{aligned} \tag{2.1}$$

where the sum over the indices μ, ν in the action and over the index a is understood. Here λ^a are hermitean generators of the Lie algebra and

$$\Psi(x,y;A) = P \exp i \int_{C_{(x,y)}} A_\mu dx_\mu \tag{2.2}$$

$\Psi(x,x;A)$ is the monodromy matrix of the connection A_μ along the closed loop $C(x,x)$ based at the point x, i.e. $\Psi(x,x;A)$ is the Wilson loop. For the group $\mathrm{SU}(N)$ using the identity:

$$\lambda^a_{\alpha\beta} \lambda^a_{\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} \tag{2.3}$$

we get:

$$\begin{aligned} 0 = \int DA_\mu \exp\Bigg(&- \frac{N}{4g^2} \int \operatorname{Tr} F_{\mu\nu}^2 d^4x \Bigg) \times \\ &\times \Bigg(\operatorname{Tr}\left(\frac{N}{g^2} D_\mu F_{\mu\nu}(z) \Psi(x,x;A) \right) + \\ &\quad + i \int_{C(x,x)} dy_\nu \delta^{(d)}(z-y) \Big(\operatorname{Tr}(\Psi(x,y;A)) \operatorname{Tr}(\Psi(y,x;A)) - \\ &\qquad - \frac{1}{N} \operatorname{Tr}(\Psi(x,y;A)\Psi(y,x;A)) \Big) \Bigg) \end{aligned} \tag{2.4}$$

where the last term vanishes in the large-N limit. The first term is the classical contribution to the loop equation, while the second term is the quantum contribution. From a modern point of view it is convenient to rephrase the loop equation into an algebraic language [17], since in this way a more powerful view of what the problem is and of its difficulty is obtained. Using the factorisation of gauge invariant operators [18] in the large-N limit and noticing that the expectation value can be combined with the matrix trace to define a new generalised trace τ, our problem is to find a (unique) solution $A_\mu(x)$ to

$$0 = \frac{1}{g^2} \tau(D_\mu F_{\mu\nu}(z) \Psi(x,x;A)) + i \int_{C(x,x)} dy_\nu \delta^{(d)}(z-y) \tau(\Psi(x,y;A)) \tau(\Psi(y,x;A)) \tag{2.5}$$

JHEP06(2005)042

for every closed contour C, with values in a certain operator algebra with normalised ($\tau(1) = 1$) trace τ [19, 7, 20]. Such solution is named the master field [21]. Thus the trace τ acts on a type II_1 von Neumann algebra [17] generated by the monodromy operator $\Psi(x, x; A)$ of Wilson loops based at x. This is the loop algebra, that is a representation of the homology algebra of based loops at x. Unfortunately, even in the case in which $A_\mu(x)$ is gaussian, this algebra is not hyper-finite at $N = \infty$, that is, it is not the limit of a sequence of finite dimensional matrix algebras, being a Cuntz algebra with an infinite number of generators [22 − 24], which is algebraically isomorphic to a free group factor with an infinite number of generators [6]. In the non-gaussian case [19, 7, 20] there is no reason for which things should be easier as to the hyper-finiteness property. The point of view pursued in this paper is that we should abandon, at first stage, the idea of solving the loop equation defined over the entire loop algebra. We should rather solve the following problem, whose solution still conveys a lot of physical information, but that is algebraically considerably simpler. For any fixed (self-avoiding, planar) based loop, $C(x, x)$, we consider the von Neumann algebra generated by $\Psi(x, x; A)$ only, which we want to determine from the loop equation restricted to $C(x, x)$. This is a commutative von Neumann algebra, for which, as it is well known, it there exists a structure theorem that is equivalent to measure theory plus spectral theory of self-adjoint operators [25]. In particular every commutative von Neumann algebra is type I and thus hyper-finite [26]. The trace on such algebra is a measure determined by the distribution of the eigenvalues $\rho_C(\lambda)$, counting multiplicity [25]:

$$\tau(\Psi(x, x; A)) = \int \exp(i\lambda)\rho_C(\lambda)d\lambda\,. \tag{2.6}$$

Thus our complicated algebraic problem reduces to finding the distribution of eigenvalues for any given fixed (self-avoiding, planar) based loop $C(x, x)$. It should perhaps be repeated that now both the trace and the (commutative) algebra are hyper-finite, so that the trace τ, contrary to the original problem, is completely known as the large-N limit of the normalised finite dimensional matrix trace $\frac{1}{N}\,\mathrm{Tr}_N$. In this respect we should mention that, already many years ago, the Migdal-Makeenko equation for any given loop was written in terms of the distribution of the eigenvalues in a way that makes (implicit) use of the algebraic commutative structure associated to iterating a given loop [8]. For self-avoiding loops in $d = 2$ in the weak coupling phase on a sphere and for small g^2A the explicit answer for the distribution of the eigenvalues is the formula mentioned in the introduction. In this paper we develop methods to solve for the distribution of the eigenvalues in $d = 2$ and $d = 4$ directly from the loop equation for planar self-avoiding loops. As mentioned in the introduction our basic idea is holography: we would like to map holographically the boundary loop equation into an equivalent bulk holographic effective action, whose critical equation determines the distribution of eigenvalues, ρ_C, via the eigenvalues of the curvature of a (twisted) local system. It may be guessed that in doing so the Cauchy theorem will play a key role, being a case (rather spectacular) of planar holography "ante litteram". We manage to change variables in the loop equation in such a way that in the new variables the quantum contribution be a regularized residue, loop independent for self-avoiding loops. This has been achieved in [9], changing variables in such a way that functional differentiation of the monodromy in the new variables, that is the basic operation to get the loop equation, produces the contour integral of a Cauchy kernel instead of a delta-like contact term. This makes us a first step closer to find an equivalent holographic quantum effective action.

The second step, by the way, following the $d = 2$ analogy of a critical equation that determines the distribution of eigenvalues, is to make the quantum contribution vanishing in the large-N limit. This is achieved in this paper by a suitable conformal mapping: the key point is that evaluating the regularized residue does not commute with the conformal mapping. For our aim, we start with representing the partition function as an integral over microcanonical strata, introducing a suitable resolution of identity [9]. These strata are characterised by given levels of the curvature of the gauge connection. Then we change variables to a holomorphic gauge in which the curvature F acquires the form $F = \bar{\partial}(\ldots)$. The new holographic loop equation follows. The last step, to annihilate the quantum term, is a conformal map to a cuspidal fundamental domain. We now describe our procedure in more detail, proceeding in parallel in $d = 2$ and $d = 4$. The $d = 4$ case is not substantially more complicated than the $d = 2$ case from a purely holographic point of view. As a first step we would like to change variable in the functional integral from the gauge connection to the curvature. Formally this is done by means of the resolution of identity:

$$1 = \int D\mu\delta(F_A - \mu) \tag{2.7}$$

in $d = 2$ and:

$$1 = \int D\mu_{\mu\nu}\delta(F_{\mu\nu}(A) - \mu_{\mu\nu}) \tag{2.8}$$

in $d = 4$, that we prefer to write decomposing the curvature into its anti-selfdual (ASD) and self-dual (SD) parts:

$$1 = \int D\mu^-_{\mu\nu}\delta(F^-_{\mu\nu}(A) - \mu^-_{\mu\nu}) \int D\mu^+_{\mu\nu}\delta(F^+_{\mu\nu}(A) - \mu^+_{\mu\nu})\,. \tag{2.9}$$

In [9] the resolution of identity associated to the ASD constraint only was imposed for reasons that will be cleared in section 4. The SD and ASD constraints are written in two dimensional language as Hitchin equations [27, 28]. In particular the ASD constraint is interpreted as an equation for the curvature of the non-hermitean connection $B = A+D = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z}$ and a harmonic condition for the Higgs field $\Psi = -iD = -i(D_u dz + D_{\bar{u}} d\bar{z})$. A is the projection of the four dimensional hermitean connection onto the $(z = x_0 + ix_1, \bar{z} = x_0 - ix_1)$ plane of the planar loop and D is the projection of the four dimensional anti-hermitean covariant derivative onto the orthogonal $(u = x_2 + ix_3, \bar{u} = x_2 - ix_3)$ plane. In this paper we choose the following notation as far as the complex basis of differentials $dz = dx_0 + idx_1$ and derivatives $\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1})$ is concerned. Thus, for example, $A_z = \frac{1}{2}(A_0 - iA_1)$. We should notice that the observable in our loop equation is somehow adapted to the microcanonical resolution of identity and thus in $d = 4$ the Wilson loop involving the non-hermitean connection, B, is considered. Yet, since at the end we would like to compute planar Wilson loops for the hermitean connection A, a more general kind of observables is needed. Indeed in section 4 we will consider $B^\lambda = A + \lambda D$

in the limit $\lambda \to 0$, which we refer to as the unitary limit. In this section for simplicity we limit ourselves to the first case. We assume a partial Eguchi-Kawai reduction from four [29 − 34] to two dimensions [9], that implies a rescaling of the classical action by a factor of N_2^{-1} [34]. This factor is needed because the partial Eguchi-Kawai reduction re-absorbs some space-time degrees of freedom into the colour degrees of freedom. In our case a two dimensional torus is reduced to a point. If the torus is commutative, N_2 is given by $\frac{1}{(2\pi)^2}\int d^2xd^2p = \frac{\Lambda^2L^2}{(2\pi)^2}$. In the non-commutative case, instead, $N_2 = \mathrm{Tr}(1) = \sum_{n\leq N_2} 1$. The loop equation for B, in the partially reduced $d = 4$ theory, reads:

$$0 = \int DB_\alpha \exp\Big(-\frac{N}{4g^2}S_{YM}\Big)\Bigg(\mathrm{Tr}\Big(\frac{N}{4g^2}\frac{\delta S_{YM}}{\delta B_\alpha(z)}\Psi(x,x;B)\Big) - \\ - i\int_{C(x,x)} dy_\alpha \delta^{(2)}(z-y)\,\mathrm{Tr}(\Psi(x,y;B))\,\mathrm{Tr}(\Psi(y,x;B))\Bigg) \tag{2.10}$$

with $\alpha = 1, 2$, so that for these variables it is as difficult to solve as for the original four dimensional connection A_μ. After implementing in the functional integral the resolution of identity by means of the gauge orbits of the microcanonical ensemble mentioned before, we change variable in the loop equation from the connection to the corresponding curvature in the holomorphic gauge. Thus we choose for the connection $A = a_z dz + \bar{a}_{\bar{z}} d\bar{z}$ in $d = 2$ and $B = b_z dz + \bar{b}_{\bar{z}} d\bar{z}$ in $d = 4$ the gauge $\bar{a}_{\bar{z}}$=0 and $\bar{b}_{\bar{z}}$=0 respectively, performing a gauge transformation in the complexification of the gauge group. This last change of variable is made in order to compute explicitly the quantum contribution as a (regularized) residue. It is well known by the Cauchy formula that the line integral along a closed loop of a holomorphic function times the Cauchy kernel with pole not lying on the loop depends only on the winding number of the loop around the pole and on the value of the holomorphic function at the pole of the Cauchy kernel:

$$f(z)Ind_C(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{z-w}dw \tag{2.11}$$

where $Ind_C(z)$ is the winding number of C around z. The change of variables to the holomorphic gauge implies that the Cauchy kernel is generated by functionally differentiating the monodromy of the connection in the loop equation with respect to the integration variables in the functional integral. However, even if we reduce the computation of the quantum contribution to the evaluation of a line integral of a Cauchy kernel, there are two obstructions for the residue theorem to apply. The first one is that the monodromy is not a holomorphic function of its endpoints. The second one is that gauge invariance requires that the functional derivative with respect to the integration variable be taken at a point of the loop, for the result to be non-vanishing, i.e at a singularity point of the Cauchy kernel. In fact we are going to compute a regularized residue obtained as the line integral of a distribution. This solves the regularity problem and the holomorphic problem at the same time. In $d = 4$ for $B = A + i\Psi$ the microcanonical representation holds in the form:

$$Z = \int \delta(F_B - \mu)\delta(d_A^*\Psi - \nu)\exp\Big(-\frac{N}{4g^2}S_{YM}\Big)DBD\mu D\nu \tag{2.12}$$

where we have re-casted the ASD microcanonical resolution of identity in the Hitchin form of a curvature equation for B plus a harmonic condition for the Higgs field Ψ, for later convenience. The curvature, F_B, is not hermitean in general and thus we may include in the functional integral the resolution of identity for its adjoint, in such a way that the integration measure is $D\mu D\bar{\mu}$ instead of $D\mu$. Yet, since the monodromy of B is in fact a functional of μ only, omitting $D\bar{\mu}$ does not affect the loop equation for B. Our convention is that $\mu = \mu^0 + n - \bar{n}$ is the curvature of B in the basis $dx_0 \wedge dx_1$ with $\nu = n + \bar{n}$. Thus μ^0 is hermitean in this basis. Only if we choose a basis different from the real one, indices are added to μ to make the choice of basis explicit. From now on in this section we mention only the $d = 4$ case, since the $d = 2$ case is obtained trivially substituting the connection B with the connection A in $d = 2$. We now change variables to the holomorphic gauge, in which the curvature of B is given by the field μ', obtained from the equation:

$$F_B - \mu = 0 \tag{2.13}$$

by means of a complexified gauge transformation $G(x;B)$ that puts the connection $B = b+\bar{b}$ in the holomorphic gauge $\bar{b} = 0$:

$$\bar{\partial} b_z = -i\frac{\mu'}{2} \tag{2.14}$$

where $\mu' = G\mu G^{-1}$. The partition function becomes:

$$Z = \int \delta(F_B - \mu)\delta(d_A^* \Psi - \nu) \exp\left(-\frac{N}{4g^2} S_{YM}\right) DB \frac{D\mu}{D\mu'} D\mu' D\nu \tag{2.15}$$

The integral over B can now be performed and the resulting functional determinants, together with the jacobian of the change of variables to the holomorphic gauge, absorbed into the definition of Γ. Γ plays here the role of a "classical" action, since we must integrate still over the field μ'. We may call Γ the "classical" holographic action, as opposed to the quantum holographic effective action, to be found in section 4. Γ is written explicitly in section 4 in terms of functional determinants. The partition function is now:

$$Z = \int \exp(-\Gamma) D\mu' \,. \tag{2.16}$$

To realize our aim of getting the Cauchy kernel it is convenient to study the loop equation for the Wilson loop involving the connection b, thought as a functional of B corresponding to gauge transforming B into the gauge $\bar{b} = 0$. Such a gauge transformation belongs to the complexification of the gauge group and it is rather a change of variable than a proper gauge transformation. However, because of the property of the trace, for closed loops, it preserves the trace of the monodromy. This allows us to transform the loop equation thus obtained into an equation for the monodromy of B. In our derivation of the loop equation, a crucial role is played by the condition that the expectation value of an open loop vanishes. In [9] two slightly different ways of achieving the vanishing of the expectation value of open b loops were presented. We may thus derive our loop equation:

$$
\begin{aligned}
0 &= \int D\mu' \operatorname{Tr} \frac{\delta}{\delta\mu'(w)} (\exp(-\Gamma)\Psi(x,x;b)) \\
&= \int D\mu' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b) \right) - \\
&\qquad\qquad - \int_{C(x,x)} dy_z \frac{1}{2} \bar{\partial}^{-1}(w-y) \operatorname{Tr}(\lambda^a \Psi(x,y;b) \lambda^a \Psi(y,x;b)) \Bigg) \\
&= \int D\mu' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b) \right) - \int_{C(x,x)} dy_z \frac{1}{2} \bar{\partial}^{-1}(w-y) \times \\
&\qquad\qquad \times \Bigg(\operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) - \\
&\qquad\qquad\quad - \frac{1}{N} \operatorname{Tr}(\Psi(x,y;b)\Psi(y,x;b)) \Bigg) \Bigg)
\end{aligned}
\tag{2.17}
$$

that in the large-N limit reduces to:

$$
\begin{aligned}
0 = \int D\mu' \exp(-\Gamma) \Bigg(&\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b) \right) - \\
&- \int_{C(x,x)} dy_z \frac{1}{2} \bar{\partial}^{-1}(w-y) \operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) \Bigg) .
\end{aligned}
\tag{2.18}
$$

The only non-trivial case is when w lies on the loop C. In this case the loop equation can be transformed easily into an equation for B. It is clear that the contour integration in the quantum term of the loop equation includes the pole of the Cauchy kernel. We need therefore a gauge invariant regularization. We proposed in [9] several slightly different ways of regularising the Cauchy kernel, that are essentially equivalent concerning the holographic form of the loop equation. The first one consists in analytically continuing the loop equation from euclidean to minkowskian space-time. Thus $z \to i(x_+ + i\epsilon)$. This regularization has the great virtue of being manifestly gauge invariant. In addition this regularization is not loop dependent. A natural, but loop dependent, regularization of the quantum contribution to the loop equation is a $i\epsilon$ regularization of the Cauchy kernel in a direction normal to the loop. We have the two possibilities of taking the internal or the external normal. We check at the end of this section that the solution of the loop equation does not depend indeed on this choice to lowest order in perturbation theory in the $d = 2$ theory. Alternatively, as suggested in [9], we may perform a conformal mapping of the region encircled by the Wilson loop to the upper-half plane, followed by regularization by means of analytic continuation to minkowskian space-time or $i\epsilon$ regularization in a direction normal to the loop. A more sophisticated form of this regularization will be presented in the next section, where we will map conformally the region whose boundary is the loop with a marked point to a cuspidal fundamental domain. Quite interestingly we will find that the regularization of the loop equation does not commute with this conformal mapping. Indeed we will take advantage of this fact to annihilate the quantum term, after having accounted for the effects of the conformal map everywhere else. In any case the result of the $i\epsilon$ regularization of the Cauchy kernel is the sum of two distributions, the principal part plus a one dimensional delta function:

$$\frac{1}{2}\bar{\partial}^{-1}(w_x - y_x + i\epsilon) = (2\pi)^{-1}(P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x))\,. \tag{2.19}$$

The loop equation thus regularized looks like:

$$\begin{aligned} 0 = \int D\mu' \exp(-\Gamma)\Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) - \int_{C(x,x)} dy_x (2\pi)^{-1} \times \\ \times\,(P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x)) \times \\ \times \operatorname{Tr}(\Psi(x,y;b))\operatorname{Tr}(\Psi(y,x;b))\Bigg)\,. \end{aligned} \tag{2.20}$$

Being supported on open loops the principal part does not contribute and the loop equation reduces to:

$$\begin{aligned} 0 = \int D\mu' \exp(-\Gamma)\Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) + \\ + \int_{C(x,x)} dy_x \frac{i}{2}\delta(w_x - y_x)\operatorname{Tr}(\Psi(x,y;b))\operatorname{Tr}(\Psi(y,x;b))\Bigg)\,. \end{aligned} \tag{2.21}$$

Taking $w = x$ and using the transformation properties of the b monodromy and of $\mu(x)'$, the preceding equation can be rewritten in terms of the connection, B, and the curvature, μ:

$$\begin{aligned} 0 = \int D\mu' \exp(-\Gamma)\Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu(x)}\Psi(x,x;B)\right) + \\ + \int_{C(x,x)} dy_x \frac{i}{2}\delta(x_x - y_x)\operatorname{Tr}(\Psi(x,y;B))\operatorname{Tr}(\Psi(y,x;B))\Bigg) \end{aligned} \tag{2.22}$$

where we have used the condition that the trace of open loops vanishes to substitute the b monodromy with the B monodromy. This is our final form of the regularized euclidean loop equation (there is an analogous form in minkowskian space-time). Let us notice the sign ambiguity in the quantum contribution that depends on the choice of the sign of $i\epsilon$. Because of the product of traces, in general the quantum contribution does not have the same operator structure as the classical term. In addition the quantum contribution is loop dependent in general. However, for self-avoiding loops, the quantum contribution is just a linear topological term added to the "classical" action, Γ, provided the loop equation is interpreted in a strong sense, that is as:

$$0 = \frac{\delta}{\delta\mu(x)}\left(\Gamma + \frac{i}{2}\int d^2x \operatorname{Tr}\mu\right)\,. \tag{2.23}$$

JHEP06(2005)042

An interpretation of this kind was attempted in [9], where the central term, being topological, was thought to be essentially irrelevant or cancelled against an anomalous phase in the effective action. Unfortunately, following the interpretation of [9], we were unable to reproduce to lowest order in perturbation theory the gauge propagator. The reason is that the central term is absolutely essential to obtain the gauge propagator to lowest order from the loop equation for self-avoiding loops, if the loop equation is interpreted weakly as:

$$0 = \tau\left(\left(\frac{\delta\Gamma}{\delta\mu(x)} + \frac{i}{2}\operatorname{Tr}(1)\right)\Psi(x,x;B)\right). \tag{2.24}$$

In fact the role of the central term in getting the correct propagator is entirely analogous to the role of the contact term in the original loop equation (eq. (2.4)). Hence we might conclude that we have come to a loose end despite our sophisticated changes of variable: perhaps we have simply rewritten our loop equation in an exotic way for exotic variables. In the two next sections we will see that, thanks to some new ingredient, this is not in fact the case.

We end this section checking to lowest order that the gauge propagator does follow from the loop equation in the holographic form in $d = 2$, when it is interpreted in a weak sense. For computational convenience we consider the holographic loop equation in $d = 2$ followed by Eguchi-Kawai reduction. We interpret it in a weak sense and we expand in powers of $\mu(x) = \exp(ipx)\mu(0)\exp(-ipx)$. We get for a small loop:

$$\tau\left(\frac{L^2}{2g^2 N_2}\mu(0)^2 - \frac{1}{2}\right) = 0 \tag{2.25}$$

where the factor of $N_2^{-1} = N^{-1}$ takes into account the Eguchi-Kawai reduction from $d = 2$ to $d = 0$ and L^2 is the area of the space-time torus. A somehow intriguing feature of our computation, based on the Eguchi-Kawai reduction, is that we perform it in a way that implies (to lowest order) a hyper-finite trace. Yet this does not imply necessarily hyper-finiteness of the algebra, that would follow instead from uniform hyper-finiteness of the trace [26]. Indeed even for the solution in terms of free random variables in $d = 2$, that certainly generate a non-hyperfinite algebra, a solution admitting a hyper-finite trace has been proposed [7]. Thus if we set:

$$\tau = \lim_N \frac{1}{N}\operatorname{Tr}_N \tag{2.26}$$

we get for μ from eq. (2.25) the normalisation:

$$\mu_{ik}(x) = \frac{g}{L}\exp(i(p_i - p_k)x) \tag{2.27}$$

for $i > k$ with $\mu_{ii} = 0$ and $\mu = \mu^*$. Since from eq. (2.7) it follows, to lowest order in powers of μ, in the gauge $D_\alpha \delta A_\alpha = 0$:

$$\begin{aligned} A_z &= i\partial(-\Delta)^{-1}\mu \\ A_{\bar z} &= -i\bar\partial(-\Delta)^{-1}\mu \end{aligned} \tag{2.28}$$

we get for the propagator to lowest order:

$$\tau(A_z(x)A_{\bar z}(y)) = \frac{g^2}{4NL^2}\sum_{i\neq k}\left(\frac{1}{(p_i-p_k)^2}\exp(i(p_i-p_k)(x-y))\right)$$
$$\sim \frac{g^2}{4(2\pi)^2}\int \frac{1}{p^2}\exp(ip(x-y))d^2p \tag{2.29}$$

as it should be in the quenched theory.

3. The two dimensional case: reduction to an abelian sub-algebra

The aim of this section is to solve the holographic loop equation of the previous section by means of a certain abelianization map in the two dimensional case. The advantage of considering the two dimensional case first is that in two dimensions we already know the exact answer, so that we can test our ideas about solving the loop equation for self-avoiding loops. As mentioned in the introduction, to realize the abelianization map we need:

- a local system;
- a uniformization to a cuspidal fundamental domain;
- a regularization at the boundary cusps that amounts to a compactification of the cusps at infinity;
- a clever choice of the gauge.

The local system was introduced in [9] basically to have a mathematically well defined model of the moduli space of the master field. Yet, it turns out that the need of a local system and in particular of the associated singularities has a deeper meaning. The local system furnishes a lattice or adelic interpretation of the microcanonical localisation:

$$F_A = \sum_p \mu_p \delta^2(x-x_p) \tag{3.1}$$

in the holographic loop equation:

$$0 = \int D\mu' \exp(-\Gamma)\Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu(w)}\Psi(x,x;A)\right) -$$
$$- \int_{C(x,x)} dy_z \frac{1}{2}\bar\partial^{-1}(w-y)\operatorname{Tr}(\Psi(x,y;A))\operatorname{Tr}(\Psi(x,y;A))\Bigg). \tag{3.2}$$

From the point of view of the functional integration the curvature of the connections associated to local systems is dense in the sense of distributions in the space of curvatures. For local systems the loop equation acquires the following form:

$$0 = \int \prod_q D\mu'_q \exp(-\Gamma)\Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu_p}\Psi(x_p,x_p;A)\right) - \tag{3.3}$$
$$- \int_{C(x_p,x_p)} dy_z \frac{1}{2}\bar\partial^{-1}(x_p-y)\operatorname{Tr}(\Psi(x_p,y;A))\operatorname{Tr}(\Psi(y,x_p;A))\Bigg)$$

where now it is necessary that the marked point of the based loop coincides with a puncture to get a non-trivial loop equation. It can be checked by direct computation that integrating in the functional integral over non-abelian local systems leads the same result for the Wilson loop as in the lattice theory:

$$\begin{aligned}\tau(\Psi(A;x,x)) &= Z^{-1}\int\prod_p dg_p\,\mathrm{Tr}(g_p\exp(i\theta_p)g_p^{-1})\,\times\\ &\quad\times\exp\Big(-\sum_p\frac{N}{2g^2a^2}\,\mathrm{Tr}(\theta_p^2)\Big)\prod_p\prod_{i\neq j}|\exp(i\theta_p^i)-\exp(i\theta_p^j)|d\theta_p\end{aligned}\tag{3.4}$$

where the product over p is restricted to the lattice points internal to the loop and $a^{-2}=\frac{\Lambda}{(2\pi)^2}=\delta^{(2)}(0)$. The dg_p integrals can be easily performed leading to the correct lattice result [3] (the only difference is the Wilsonian lattice action instead of its formal continuum limit):

$$\tau(\Psi(A;x,x))=Z^{-1}\prod_p\int(\mathrm{Tr}(\exp(i\theta_p))\exp\Big(-\frac{N}{2g^2a^2}\,\mathrm{Tr}(\theta_p^2)\Big)\prod_{i\neq j}|\exp(i\theta_p^i)-\exp(i\theta_p^j)|d\theta_p$$

that is equivalent for $\tau(\Psi)$ to the formula for the distribution of eigenvalues in the introduction for small g^2A and in the scaling limit $N_C\to\infty$ with $N_Ca^2=A=$ constant, where N_C is the number of punctures inside the loop C. We will see now how this result is reproduced by the abelianization map of the holographic loop equation. The key point is that identifying the marked point of the based loop with a puncture contains implicitly the possibility of a change of the conformal structure. Formally the Wilson loop is invariant under re-parameterisation of the boundary that can be extended to conformal transformations of the region encircled by the loop, but in fact we will change the conformal structure in a way that is equivalent to attach to the loop, in a neighbourhood of x, an infinitesimal strip going to infinity. This does not alter the Wilson loop because of the zig-zag symmetry [35]. Of course the classical action in $d=2$ is not conformally invariant, so that if we are going to compute the classical action in terms of a conformally transformed local system we must transform the classical action properly. Coming back to the loop equation for local systems, it is of the utmost importance that the regularized residue with the new conformal structure vanishes in the loop equation. We now explain why, first proceeding heuristically and then by direct computation. Our lattice gives rise to a punctured sphere with a based contour, $C(x,x)$, that determines an internal region (the smaller one on a large sphere), Ω_x, with the topology of a disk with some punctures inside and at least one on the boundary, that coincides with the distinguished base point of the loop, x. Ω_x can be mapped conformally to a cuspidal fundamental domain on the upper-half plane with all the cusps but one, for example the cusp on the boundary, on the $y=0$ axis and the remaining one at $y=\infty$. On the other hand also the region external to the loop can be mapped conformally in a similar way. Heuristically this conformal map allows us to diagonalise the curvature at the cusps, thanks to a residual gauge symmetry (see below), provided the cusps are compactified. In general the compactification is not possible, otherwise every local system on a sphere would be abelian. However, in our case, the compactification is required since the marked point, x, in origin belongs to the closed loop $C(x, x)$ and thus it is not a puncture. In addition, assuming rotational invariance on the sphere, if one cusp is compactified also all the remaining ones should be. On the fundamental domain all the cusps but one are on the x-axis and hence, choosing an axial gauge along the y-axis, there is a residual gauge symmetry of making gauge transformations along the x-axis to diagonalise the curvature at the cusps on the x-axis. In addition, since our local systems define a representation of the fundamental group of a punctured sphere, also the curvature at the remaining cusp at $y = \infty$ must be diagonal. Thus we get an abelian local system. But abelianization can only be consistent with a vanishing of the regularized residue in the loop equation, since an abelian system becomes classical in the large-N limit. Thus there is no quantum term in the large-N loop equation on the cuspidal fundamental domain, and being gauge invariant, the loop equation is to coincide before and after gauge fixing. Hence there never is a quantum residue in any gauge. Notice that the abelianization could not be performed before the conformal mapping, since it is essential that the cusps be on the same line and compactified. Now we must check directly that the quantum residue vanishes in any gauge. The cuspidal domain is a polygon defined by the uniformization theory of Riemann surfaces with punctures and boundaries. A unified approach in which the boundaries are treated in a way similar to the punctures has been given by Penner [36]. Remarkably Penner approach involves the choice of at least one puncture on the boundaries, precisely as required by our interpretation of the holographic loop equation in terms of local systems. The boundary arcs of the polygon that uniformizes our region Ω_x are oriented in the following way. The couple of arcs that end into a cusp corresponding to the punctures in the interior of Ω_x have opposite orientations on the fundamental domain, since they are in fact identified by the gluing map that reconstructs from the polygon the punctured Riemann surface: this identification creates tubes out of such cusps. As a consequence these couple of arcs share the same orientation on the reconstructed Riemann surface. On the contrary, the arcs ending into the cusp lying on the loop share the same orientation on the polygon since they are not glued together: these arcs are associated to an infinitesimal strip going to infinity. As a consequence these couple of arcs have opposite orientation on the strip to infinity. This difference in orientation plays a crucial role in evaluating the regularized residue at the internal cusps and at the boundary cusps. In fact in the first case the regularized residues associated to the two asymptotes of the cusp sum up to 1 because of the same orientation of the asymptotes on the Riemann surface. In the second case the sum is 0, because the opposite orientation of the asymptotes on the Riemann surface. We may look at the last fact as just another consequence of the zig-zag symmetry. Thus the quantum contribution vanishes. We can summarise our argument as follows. The local system is invariant under the conformal map, while the classical action transforms in a definite way. The marked point of the based loop becomes a cusp of the fundamental domain. This cusp is obtained adding an infinitesimal strip going to infinity to the loop in a neighbourhood of the marked point. Because of the zig-zag symmetry, at this cusp the quantum regularized residue vanishes. We can take into account the preceding argument to derive a new form of the holographic loop equation that leads to a critical equation for a holographic quantum effective action. The resolution of the identity involves now local systems on the cuspidal fundamental domain with coordinates $(t, \bar{t})$:

$$1 = \int \delta(F_A^{(t)} - \mu^{(t)}) D\mu^{(t)} \tag{3.5}$$

where the superscript $^{(t)}$ refers to fields defined on the fundamental domain. It is this new resolution that is inserted into the functional integral:

$$0 = \int DAD\mu^{(t)} \exp\left(-\frac{N}{2g^2}\int \mathrm{Tr}\, F_A^2 d^2x\right) \delta(F_A^{(t)} - \mu^{(t)}) D\mu^{(t)} \,. \tag{3.6}$$

Then all the steps go through as in the second section. The universal cover of the fundamental domain is the upper-half plane U. On U we choose the gauge $A_y = 0$, that leaves, as a residual gauge symmetry, gauge transformations that are y independent. In the gauge $A_y = 0$ the determinant due to localisation, i.e. the one obtained integrating with respect to the gauge connection, A, the delta functional in eq. (3.6), and the Faddeev-Popov determinant are both field independent and cancel each other. We can use the residual gauge symmetry to fix the gauge $\mu_p^{ch} = 0$ (the label ch means the non-diagonal part) at the cusps on the x-axis. The associated extra Faddeev-Popov determinant is the square of the Vandermonde determinant of the eigenvalues of the curvature of the local system at the punctures. Then the holographic quantum effective action reduces to:

$$\Gamma_q = \frac{N}{2g^2a^2}\sum_i\sum_p |\frac{\partial t}{\partial z}(p)|^2 h_p^{(t)i2} - \sum_{i\neq j}\sum_p log|h_p^{(t)i} - h_p^{(t)j}| - \log \mathrm{Det}\left(\frac{Dh^{(t)}}{Dh^{(t)\prime}}\right) \tag{3.7}$$

where the $^{(t)}$ superscript refers to the domain of definition of the lattice field, h_p, and we have set $a = 2\pi/\Lambda$, with a the lattice spacing corresponding to the cutoff Λ of the theory, that comes from the product of delta functions at the same point in the classical action. The last term is the logarithm of the jacobian to the holomorphic gauge of the abelian local system. However it vanishes identically in a gauge in which the curvature is abelian (see at the end of next section). It should be noticed that Γ_q is expressed as a functional of the local system on the fundamental domain. This involves in the classical term a change of the metric since the classical action is not conformally invariant. Since on the fundamental domain the quantum term vanishes, the loop equation for self-avoiding loops reduces to:

$$0 = \tau\left(\frac{\delta\Gamma_q}{\delta h_p^{(t)}}\Psi(x_p, x_p; A)\right) . \tag{3.8}$$

Eq. (3.8) is implied by the critical equation:

$$\frac{\delta\Gamma_q}{\delta h_p^{(t)}} = 0 \,. \tag{3.9}$$

Getting a saddle-point morally implies that we have somehow integrated away the order of N^2 non-abelian degrees of freedom to obtain the effective action for the order of N remaining eigenvalues. It is the conformal map that allowed us to diagonalise the curvature

JHEP06(2005)042

at the cusps. But this works only for a system in which the curvature has delta-like singularities. Hence local systems know about quantum field theory ! It is perhaps this the very reason for the occurrence of Hitchin systems in $d = 4$ quantum field theories (see section 4 and section 7). Notice that the abelianization works only for self-avoiding loops. In case of self-intersection the marked point would be ramified and globally there would not be enough residual symmetry in an axial gauge on a ramified covering to diagonalise all the cusps. On the other hand the regularized residue in the loop equation does not vanish in general at a ramification point. It remains to be seen that our critical equation coincides with the large-N saddle point in eq. (1.1). To do so we need a better understanding of the uniformization map. It turns out that the relevant mathematics is still the one of the moduli space of bordered Riemann surfaces following [36]. As already noticed, Penner version of the Teichmuller theory of bordered surfaces involves the choice of at least a puncture and a horocycle arc in a neighbourhood of the puncture on each boundary of the surface, in analogy with the theory of punctured surfaces that involves a horocycle around each internal puncture. More generally we may consider the case in which the loop in the loop equation intersects a number of punctures of our lattice. It is clear that the preceding arguments about the vanishing of the quantum residue apply in this more general situation, since they depend indeed on the local structure around each puncture. But now, since all the punctures on the loop are attached to infinitesimal strips going to infinity, the circle at infinity contains a lattice (or adelic) image of the original loop through the conformal map. Quadratic differentials [37] can be used to construct the uniformization map from the generic region that occurs in the loop equation to the cuspidal fundamental domain. The basic relation between quadratic differentials, q, and the uniformization map is:

$$\frac{\partial t}{\partial z} = \sqrt{q}\,. \tag{3.10}$$

We need therefore the standard form of a quadratic differential near a cusp:

$$\frac{\partial t}{\partial z} = \frac{L}{2\pi i z} \tag{3.11}$$

where L is the length of the horocycle arc around the cusp. Since this expression is infinite at the cusps it must be regularized and suitably interpreted. In particular it depends crucially on what the cutoff is on the fundamental domain near the cusps. This may be difficult to understand in general, but it seems easier in the case of a circular loop. In this case we have essentially a circle that is mapped (adelically) into the circle at infinity. This is a cylinder, i.e. a punctured disk, which is mapped by the uniformization map to a strip in the upper-half plane. In this case the uniformization map is:

$$t = \frac{L}{2\pi i}\log(z)\,. \tag{3.12}$$

Thus we get:

$$\left|\frac{\partial t}{\partial z}\right|(p)^2 = \frac{R^2}{a^2} = \frac{A}{\pi a^2} \sim N_C \tag{3.13}$$

JHEP06(2005)042

where R is the radius and A the area of the disk, while a is the radius of a little disk around the puncture. Thus N_C is the number of lattice points inside the disk. When eq. (3.13) is inserted into eq. (3.7) the correct distribution of eigenvalues, given by the saddle point equation for the effective action in eq. (1.1), is obtained after noticing that θ^i in eq. (1.1) is related to $h^i_p = h^i$ (by rotational invariance on the sphere) in eq. (3.7) by:

$$\theta^i = N_C h^i \tag{3.14}$$

since the magnetic flux through the loop counts the number, N_C, of punctures inside. In fact the holographic action in terms of θ^i becomes $\sum_p(\frac{N}{2g^2 N_C a^2}\sum_i \theta^{i2} - \sum_{i\neq j} log|\theta^i - \theta^j| + \text{constant})$ which differs by an irrelevant overall factor and an additive constant from the one in eq. (1.1). In this paper we have ignored the contribution of the region external to the loop and the corresponding conformal map. We will consider it elsewhere. It should be noticed that the eigenvalues of the curvature at the punctures of the local system are in fact defined modulo 2π. We can take into account the periodicity of h_p summing over appropriate windings. This is expected to lead to the known phase transition for $g^2 A$ large in the large-N limit [3−5], but in that case the external region must be considered too.

4. The four dimensional case: reduction to a Borel sub-algebra

The key points in the previous section can be extended to the planar four dimensional case. We list here the needed modifications. We should remind that in our approach the observable in the loop equation is adapted to the microcanonical resolution of identity. In the four dimensional case the resolution of identity involves the SD or ASD curvature. Thus, for example in the ASD sector, the connection that enters the loop equation is $B = A + D$, which is non-unitary. However for physical reasons we are interested to compute the Wilson loop for the unitary connection A. Thus we introduce $B^\lambda = A + (\lambda D_u dz + \bar{\lambda} D_{\bar{u}} d\bar{z})$ where λ is a section (possibly constant) of a holomorphic line bundle and we take the limit $\lambda \to 0$. We refer to this limit as the limit of unitary Wilson loop or the unitary limit in short. In the four dimensional case it is most convenient to take the space-time to be a product of a two dimensional sphere by a non-commutative torus in the limit of infinite non-commutativity [9]. This limit is known to be equivalent to the usual commutative theory in the large-N limit [31]. Although several different space-times may be considered, in this paper we make this choice because the associated SD or ASD equations look like a kind of infinite dimensional vortex equations (see below). Heuristically we find appealing the occurrence of vortices for the hope of reproducing both the area law at large distances and the Coulomb law at short distances. The choice of the sphere is also modelled on the analogy with the two dimensional case, especially for simplifications due to the non-existence of non-trivial cycles, but for the ones associated to the punctures. Indeed on a sphere all the moduli of the local system are the local moduli. When we take the unitary limit in our loop equation, some care is necessary, since the limit has to be taken in such a way that the correct four dimensional information survives, for example in the beta function. The first coefficient of the beta function (see next section) can be computed in two different but related ways, that we now explain. Since the theory lives on a product of a sphere by a non-commutative torus, the curvature equation involves a central term, H, equal to the inverse of the parameter of non-commutativity, θ. This occurs because, once the gauge connection is required to vanish at infinity up to gauge equivalence, the only term that survives in the curvature at infinity is the commutator of the derivatives on the non-commutative torus, that is H. In turn H vanishes as $\frac{1}{N}$ in the large-N limit. Therefore, in the case of B^{λ}, our centrally extended and λ twisted ASD curvature equation reads:

$$\begin{aligned} F_A - i|\lambda|^2\Psi^2 &= \sum_p \mu_p^0 \delta^{(2)}(x - x_p) + H1 \\ \lambda\bar{\partial}_A\psi &= \sum_p \lambda n_p \delta^{(2)}(x - x_p) \\ \bar{\lambda}\partial_A\bar{\psi} &= \sum_p \bar{\lambda}\bar{n}_p \delta^{(2)}(x - x_p) \end{aligned} \tag{4.1}$$

where we have set $D = i\Psi$, we have rescaled n by a factor of λ to ensure the finiteness of ψ in the $\lambda \to 0$ limit and analogously in the SD case. The central extension H is referred to in this paper as the twist of the local system. The λ rescaling instead is a twist of the Hitchin system considered as a twisted Higgs bundle. In our interpretation of the preceding equations as vortex equations, the central extension H is related to the non-vanishing of the Higgs field at infinity, while the zeroes of the Higgs field may arise from twisting by the factor of λ. In fact this is precisely what we require in the unitary limit. We choose a λ that has a lot of zeroes in a compact set containing the loop C, in order to make the Higgs field vanishing small in a neighbourhood of the loop to ensure unitarity of the monodromy along C, and that converges to 1 at infinity, to keep the information about the four dimensional nature of the theory. As to the beta function, the simplest case is $\lambda = 1$. In this case, if we are interested only in the ultraviolet logarithmic divergences for the collective field μ^0 and not in the critical equation for the holographic quantum effective action, we need not to decompose the collective field μ^0 into a sum of delta distributions, that is equivalent to introduce the local system, and in fact β_0 can be found directly from the ultraviolet divergences of the "classical" holographic action, Γ, looking at the $\mathrm{Tr}(\mu^{02})$ counter-term (see next section), solving in perturbation theory, at first order in power of the local curvature, the preceding equation around the $\mu^0 = n = 0$ solution. However our observable in the case $\lambda = 1$ is not physically interesting. In addition it is trivial at least at first order in perturbation theory and in the large-N limit. This occurs because the contribution of the propagator of the field A_z is exactly cancelled by the one of A_u because of the different factors of i in the Wilson loop. In fact at $\lambda = 1$ the Wilson loop is probably topological in the large-N limit. The limit $\lambda \to 0$ is the most physically interesting. In this limit H must be left untouched at infinity since it is essential to keep the correct four dimensional information while n is expected to be rescaled by a factor of λ. A little thought shows that this may happen if λ has a zero at each puncture in a compact set containing the loop and converges to 1 at infinity as already anticipated. This implies that B^{λ} converges to $(A_z + \partial_u)dz + (A_{\bar{z}} + \partial_{\bar{u}})d\bar{z}$ at infinity and to A on a compact set on the sphere in the complement of infinity. In this case it is more convenient the lattice or adelic interpretation of the curvature equation in order to obtain β_0. It is natural to interpret the adelic theory as defining a scaling limit as in usual lattice gauge theories. In such theories there are no ultraviolet divergences because of the finite lattice spacing, but all the divergences appear as logarithmic infrared divergences at a scale much larger than the lattice spacing but still smaller than the inverse of Λ_{QCD}, that goes to infinity in the scaling limit, i.e. when the coupling constant goes to zero. If we are interested in the infrared logarithmic divergence we should employ in our computations the asymptotic value of λ at infinity, i.e. 1, and hence our computation reduces to the one in the case $\lambda = 1$ everywhere. For our twisted local system the region of asymptotic freedom is thus the large distance region on the sphere. This perhaps resembles the holographic UV-IR duality already encountered in [10]. From a technical point of view, as far as β_0 is concerned, the computation for $\lambda = 1$ and the one for $\lambda \to 0$ are identical once it is observed that the in the second case we are actually looking at a neighbourhood of infinity. Thus in this paper we report only the first case in the next section. We have seen that having introduced non-commutativity leads inevitably to a central extension in the curvature equation and thus for $\lambda = 1$ we get a central extension of an infinite dimensional representation of the fundamental group of the punctured Riemann surface. This central extension cannot be decomposed in general into a U(1) abelian system and a flat connection as in finite dimensions. Thus we may wonder which are the finite dimensional approximations of this system. The answer is quiver bundles with parabolic singularities [38]. In $d = 4$ the additional difficulty arises that is not possible to impose the two SD and ASD constraints independently at the same time for an irreducible connection in the functional integral. Thus either we impose only the ASD part of the resolution of identity as we did in [9] (we are not finding a universal master field, since our observable is adapted to the resolution of identity) or we introduce orbifold models as follows. There are orbifold models whose large-N limit is equivalent to the large-N limit of QCD. We explain what a orbifold is in this context following [39]:" A certain parent gauge theory is chosen", in our case pure SU(N) gauge theory; "the orbifold theory is simply given throwing away all fields that are not invariant under a discrete subgroup of the gauge symmetry " in our case Z_2. "The resulting theory has the remarkable property that at large-N its perturbation series is the same as the parent theory, up to some simple rescaling of N". Sometimes, using the loop equation, it is even possible to show non-perturbative equivalence between the parent and orbifold theory. This is the case for our Z_2 orbifold, that is simply a gauge theory with gauge group SU(N) $\times$ SU(N). It is clear that if we compute the Wilson loop in the $(N, \bar{N})$ representation in the Z_2 orbifold theory in the large-N limit, this will be the same as the Wilson loop in the adjoint representation of the parent SU(N) theory. As we said the necessity of a Z_2 orbifold occurs in $d = 4$, if we want to impose in the functional integral the two SD and ASD constraints independently at the same time. We now write the loop equation in terms of the lattice field of curvatures in $d = 4$:

$$0 = \int \prod_q D\mu'_q \exp(-\Gamma) \left(\operatorname{Tr} \left(\frac{\delta \Gamma}{\delta \mu_p} \Psi(x_p, x_p; B) \right) - \right. \tag{4.2}$$

$$- \int_{C(x_p,x_p)} dy_z \frac{1}{2}\bar{\partial}^{-1}(x_p - y)\,\mathrm{Tr}(\Psi(x_p, y; B))\,\mathrm{Tr}(\Psi(y, x_p; B))\Big)$$

and analogously in the unitary limit:

$$\begin{aligned} Z &= \int \delta(F_{B^\lambda} - \mu^\lambda - H1)\delta(d^*_{A^\lambda}\Psi^\lambda - \nu^\lambda)\exp\Big(-\frac{N}{4g^2}S_{YM}\Big)DB\frac{D\mu^\lambda}{D\mu'^\lambda}D\mu'^\lambda D\nu^\lambda \\ &= \int \exp(-\Gamma^\lambda)D\mu'^\lambda \end{aligned} \tag{4.3}$$

where a λ dependent effective action appropriate for the study of the monodromy of the operator $B^\lambda = A + i(\lambda\psi + \bar{\lambda}\bar{\psi}) = A^\lambda + i\Psi^\lambda$, with $\mu^\lambda_{z\bar{z}} = \mu^0_{z\bar{z}} + \lambda n_{z\bar{z}} - \bar{\lambda}\bar{n}_{z\bar{z}}$, must be introduced. As in $d = 2$, to get rid of the quantum term, we must map the lattice loop equation to a cuspidal fundamental domain. In doing so we make a conformal rescaling of the Hitchin system as dictated by the conformal properties induced requiring the conformal invariance of the non-unitary monodromy that enters the loop equation:

$$B_z = \frac{\partial t}{\partial z}B_t\,. \tag{4.4}$$

Thus, in mapping to the fundamental domain, we are actually rescaling in a certain way also H, the central extension, and the orthogonal coordinates as well. Now, to get a zero quantum contribution, we must insert in the functional integral the Wilson loop and the resolution of identity, expressed in terms of the Hitchin system on the fundamental domain (we do not display explicitly the dependence on λ):

$$\begin{aligned} Z &= \int \delta(F^{(t)}_B - \mu^{(t)} - H^{(t)}1)\delta((d^*_A\Psi)^{(t)} - \nu^{(t)}) \times \\ &\quad \times \exp\Big(-\frac{N}{4g^2}S_{YM}\Big)DB\frac{D\mu^{(t)}}{D\mu'^{(t)}}D\mu'^{(t)}D\nu^{(t)} \\ &= \int \exp(-\Gamma)D\mu'^{(t)}\,. \end{aligned} \tag{4.5}$$

As in $d = 2$ the Wilson loop is invariant by construction, up to the added infinitesimal strip to infinity, that is attached and got rid by the zig-zag symmetry. The classical action must be expressed in terms of the Hitchin system on the fundamental domain as in the $d = 2$ theory:

$$S_{YM} = \frac{1}{N_2}\int \frac{d^2z}{4}\frac{2\pi}{H_{z\bar{z}}}\mathrm{Tr}[\ldots](z,\bar{z}) = \frac{1}{N_2}\int \frac{d^2t}{4}\frac{2\pi}{H_{t\bar{t}}}\mathrm{Tr}[\ldots](t,\bar{t})\,. \tag{4.6}$$

The contribution to S_{YM} in the ASD sector maintains the same form because of its invariance under conformal rescaling of the metric. For the SD part there might be additional terms due to the fact that a holomorphic twist of the Higgs field in the ASD sector is anti-holomorphic in the SD one. The holographic quantum effective action reads (we do not display explicitly the dependence on λ):

$$\Gamma_q = \Gamma|_{A_y=0}|_{FD} - \sum_p\sum_{i\neq j}\log\mathrm{Det}(ad\mu^+_p)|_{\mu^-_p=0}\,. \tag{4.7}$$

The first term, $\Gamma|_{A_y=0}|_{FD} = ((\frac{N}{4g^2}S_{YM} + \frac{1}{2}\log \mathrm{Det}'(-\Delta_A\delta_{\mu\nu} + D_\mu D_\nu + iad_{F^-_{\mu\nu}}) - \log \mathrm{Det}' \times \frac{D\mu}{D\mu'})|_{A_y=0})|_{FD}$, is the "classical" holographic action associated to the reduced non-commutative theory on the fundamental domain in the axial gauge. From a computational point of view it is more convenient to calculate the functional determinants in terms of the original Hitchin system before the conformal map to the fundamental domain. The two expressions for Γ should coincide up to the conformal anomaly. Thus $\Gamma|_{A_y=0}|_{FD} = \Gamma|_{A_y=0}|_{\mathrm{Sphere}}$ + Conformal Anomaly. The structure of $\Gamma|_{A_y=0}|_{\mathrm{Sphere}}$ will be elucidated below. The second term is due to the additional gauge fixing to a Borel sub-algebra, $\mu_p^- = 0$, and it is the $d = 4$ analogue of the Vandermonde determinant of the eigenvalues, to which it reduces in the unitary limit $\lambda \to 0$. The label $^-$ for μ_p means here lower triangular part, excluding the diagonal, while the label $^+$ for μ_p means here upper triangular part including the diagonal. The conformal anomaly is the contribution due to the change of metric implicit in mapping conformally the region encircled by the punctured Wilson loop to the cuspidal fundamental domain. It can be obtained from the exact beta function. The holographic loop equation on the fundamental domain reads (we do not display explicitly the dependence on λ):

$$0 = \tau\left(\frac{\delta\Gamma_q}{\delta\mu_p^{+(t)}}\Psi(x_p, x_p; B)\right) \tag{4.8}$$

that is implied by the critical equation:

$$\frac{\delta\Gamma_q}{\delta\mu_p^{+(t)}} = 0 \tag{4.9}$$

which we refer to as the master equation. We should notice that, in analogy with the two dimensional case, corrections due to the external region and to the periodicity of the eigenvalues should be included. We write now the "classical" holographic action, Γ, in terms of functional determinants. For the purpose of computing the first coefficient of the beta function it is considerably more convenient to calculate Γ in a Lorentz gauge rather than Γ_q in the axial gauge, that is needed to realize the reduction to a Borel sub-algebra and, in the unitary limit, to a diagonal one. We expect that the divergences of Γ and Γ_q coincide, since Γ_q is obtained from Γ just by a conformal map and a suitable gauge fixing. Since however the conformal map is singular, subtleties might be involved and thus we would like to have a direct understanding of Γ_q. Yet, we will not address this problem in this paper. Part of the computation of Γ in a Lorentz gauge appeared already in [9]. However we have computed here more explicitly than in [9] the jacobian of the change of variables to the holomorphic gauge. This jacobian turns out to be essential to reproduce the correct value of β_0 in the limit $\lambda \to 0$, which is the one relevant to the physical case of Wilson loops for the unitary connection A. In this respect the computation of the beta function that we present here is more relevant to the physics than the one in [9], where the jacobian to the holomorphic gauge was not computed and SD of the master field implicitly assumed in the sector of ASD type. In this case it was found in [9] that the one-loop perturbative beta function was accounted completely by the localisation determinant (see next section). In the Feynman gauge Γ is the sum of the classical (reduced, non-commutative) Yang-Mills action plus the logarithm of a number of determinants whose origin is as follows. There is the determinant that arises from the localisation integral on the microcanonical ensemble. The localisation determinant is defined formally as:

$$\begin{aligned}\int DA_\mu \delta(F^-_{\mu\nu} - \mu^-_{\mu\nu}) &= \mathrm{Det}'^{-1}(P^- d_A \wedge) \\ &= \mathrm{Det}'^{-\frac{1}{2}}((P^- d_A\wedge)^*(P^- d_A\wedge)) \\ &= \mathrm{Det}'^{-\frac{1}{2}}(-\Delta_A \delta_{\mu\nu} + D_\mu D_\nu + i ad_{F^-_{\mu\nu}})\end{aligned} \tag{4.10}$$

where P^- is the projector onto the anti-selfdual part of the curvature. The $'$ suffix requires projecting away from the determinants the zero modes due to gauge invariance, since gauge fixing is not implied in the loop equation, though it may be understood if we like to.

The careful reader may have noticed the unusual spin term, $i ad_{F^-_{\mu\nu}}$, as opposed to $2 i ad_{F_{\mu\nu}}$ which arises in the perturbative effective action (see next section). The occurrence of $F^-_{\mu\nu}$ is due to the projector P^- in eq. (4.10). The coefficient of $F^-_{\mu\nu}$ is one half of the one of $F_{\mu\nu}$ since the delta functional in eq. (4.10) is approximated by a gaussian whose quadratic form is the second order expansion of $(F^-_{\mu\nu} - \mu^-_{\mu\nu})^2$ around $\mu^-_{\mu\nu}$, while the usual background field method involves expanding the quadratic form associated to $F^2_{\mu\nu}$ around the background $\mu_{\mu\nu}$ (see next section). After inserting the gauge fixing condition and the corresponding Faddeev-Popov determinant the localisation determinant can be defined non-formally in the Feynman gauge as:

$$\begin{aligned}\mathrm{Det}'^{-\frac{1}{2}}(-\Delta_A \delta_{\mu\nu} + D_\mu D_\nu + i ad_{F^-_{\mu\nu}}) &= \lim_{\epsilon\to 0} \int DcDA_\mu \exp\left(-\frac{1}{2\epsilon}\int d^4x\, \mathrm{Tr}(c^2)\right) \times \\ &\quad \times \exp\left(-\frac{1}{4\epsilon}\sum_{\mu\neq\nu}\int d^4x\, \mathrm{Tr}((F^-_{\mu\nu} - \mu^-_{\mu\nu})^2)\right) \times \\ &\quad \times \delta(D_\mu \delta A_\mu - c)\Delta_{FP}\,.\end{aligned} \tag{4.11}$$

The result is thus:

$$\mathrm{Det}'^{-\frac{1}{2}}(-\Delta_A \delta_{\mu\nu} + D_\mu D_\nu + i ad_{F^-_{\mu\nu}}) = \mathrm{Det}^{-\frac{1}{2}}(-\Delta_A \delta_{\mu\nu} + i ad_{F^-_{\mu\nu}})\Delta_{FP} \tag{4.12}$$

where $F^-_{\mu\nu}$ is the anti-selfdual part of the field strength, given by:

$$F^-_{\mu\nu} = F_{\mu\nu} - F^*_{\mu\nu} \tag{4.13}$$

with:

$$F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}\,. \tag{4.14}$$

In addition the determinant that arises as the jacobian of the change of variables to the holomorphic gauge contributes to the "classical" holographic action:

$$-\log \mathrm{Det}' \frac{D\mu}{D\mu'} \tag{4.15}$$

JHEP06(2005)042

which can be computed more explicitly as:

$$\log \mathrm{Det}\left(1+\left[G^{-1}\frac{\delta G}{\delta \mu},\mu\right]\right) \tag{4.16}$$

where the following equation has been used:

$$G^{-1}\bar{\partial}G=-i\bar{b}\,. \tag{4.17}$$

The "classical" holographic action, Γ, reads:

$$\Gamma=\frac{N}{4g^2}S_{YM}+\frac{1}{2}\log\mathrm{Det}(-\Delta_A\delta_{\mu\nu}+iad_{F^-_{\mu\nu}})-log\Delta_{FP}-\log\mathrm{Det}\,\frac{D\mu}{D\mu'} \tag{4.18}$$

where S_{YM} is the Yang-Mills action of the reduced non-commutative theory and we assume a corresponding reduction in the other terms where necessary. The normalisation of Γ given by the reduced theory is needed to be compatible with the two dimensional normalisation of the Vandermonde-like determinant in eq. (4.7). Notice that the last term in eq. (4.18) vanishes if μ is reduced by gauge fixing to an abelian or to a Borel sub-algebra, since the commutator in eq. (4.16) involves also the colour indices and the commutator of elements in the Borel sub-algebra is nilpotent.

5. Beta function: the first coefficient

We compute in this section the first coefficient of the beta function as it follows from the "classical" holographic effective action Γ. In this section for simplicity we evaluate Γ in the un-reduced theory, rather than in the reduced one. The reduction is irrelevant as far as β_0 is concerned, since its computation does not involve the second term in eq. (4.7). A partial computation involving only the part of Γ composed by the classical Yang-Mills action and the localisation determinant was already performed in [9]. Following [9], it is convenient to perform the computation in an indirect way, by means of a term by term comparison with the usual one-loop perturbative contribution to the effective action. For this purpose let us recall the structure of one-loop perturbative corrections to the classical action in the Feynman gauge:

$$\begin{aligned}\int DcDA_\mu \exp\Big(-\frac{N}{2g^2}\int d^4x\,\mathrm{Tr}(c^2)\Big)\exp\Big(-\frac{N}{4g^2}S_{YM}\Big)\delta(D_\mu\delta A_\mu-c)\Delta_{FP}=\\ =\exp\Big(-\frac{N}{4g^2}S_{YM}\Big)\mathrm{Det}^{-\frac{1}{2}}(-\Delta_A\delta_{\mu\nu}+i2ad_{F_{\mu\nu}})\,\mathrm{Det}(-\Delta_A)\end{aligned} \tag{5.1}$$

where we have inserted in the functional integral the gauge-fixing condition and the corresponding Faddeev-Popov determinant and, by an abuse of notation, we have denoted with A the classical background field in the right hand side of eq. (5.1). It follows that the perturbative one-loop effective action in the Feynman gauge is given by:

$$\Gamma_{one-loop}=\frac{N}{4g^2}S_{YM}+\frac{1}{2}\log\mathrm{Det}(-\Delta_A\delta_{\mu\nu}+i2ad_{F_{\mu\nu}})-\log\mathrm{Det}(-\Delta_A)\,. \tag{5.2}$$

The perturbative computation of the one-loop beta function [40, 41] is the result of two contributions that are independent within logarithmic accuracy [42]. The orbital contribution gives rise to diamagnetism and to a positive term in the beta function:

$$\begin{aligned} -\log(\mathrm{Det}^{-\frac{1}{2}}(-\Delta_A\delta_{\mu\nu})\,\mathrm{Det}(-\Delta_A)) &= \log\mathrm{Det}(-\Delta_A) \\ &= \frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)\frac{1}{2}\sum_{\mu\neq\nu}\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2 \end{aligned} \tag{5.3}$$

where it should be noticed the cancellation of two of the four polarisations between the first factor and the Faddev-Popov determinant. The spin contribution gives rise to paramagnetism and to an overwhelming negative term in the beta function [42]:

$$\begin{aligned} \frac{1}{4}\sum_{\mu\neq\nu}\mathrm{Tr}(i2ad_{F_{\mu\nu}}(-\Delta_A)^{-1}i2ad_{F_{\mu\nu}}(-\Delta_A)^{-1}) &= 2\,\mathrm{Tr}(iad(\mu^0)(-\Delta_A)^{-1}iad(\mu^0)(-\Delta_A)^{-1}) \\ &= -\frac{12}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)\frac{1}{2}\sum_{\mu\neq\nu}\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2 \end{aligned} \tag{5.4}$$

where for later convenience we have expressed the spin contribution in the $\lambda \to 0$ limit in terms of the field μ^0. Hence the complete result for the divergent part of $\Gamma_{one-loop}$ is:

$$\left(\frac{N}{2g^2} - \frac{11}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)\right)\frac{1}{2}\sum_{\mu\neq\nu}\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2 \tag{5.5}$$

from which it follows that:

$$\beta_0 = \frac{11}{12}\frac{1}{(2\pi)^2}\,. \tag{5.6}$$

From eq. (4.18) it can be easily read that the orbital contribution in $\Gamma_{one-loop}$ and Γ coincide. On the contrary, the spin contribution in Γ involves only the anti-selfdual part of the curvature instead of the curvature itself. Hence the spin contribution from the localisation determinant in the unitary limit is only one half of the spin contribution in perturbation theory. Thus the orbital and spin contributions of the localisation determinant sum up to:

$$\left(\frac{N}{2g^2} - \left(\frac{1}{2} - \frac{1}{12}\right)\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda}{\mu}\right)\right)\mathrm{Tr}(\mu^{02})\,. \tag{5.7}$$

In fact, remarkably, the missing other one half in the coefficient of the beta function is furnished by the jacobian to the holomorphic gauge in the unitary limit, which to second order in μ^0 contributes to Γ:

$$\log\mathrm{Det}\left(1 - i\left[\bar\partial^{-1}\left.\frac{\delta\bar b}{\delta\mu}\right|_{\mu=\mu^0}, \mu^0\right]\right) \tag{5.8}$$

where the commutator involves space-time indices and colour as well. From the *ASD* localisation it follows to lowest order in μ:

$$\frac{\delta\bar b}{\delta\mu} = -i\bar\partial(-\Delta)^{-1} + \bar\partial_u(-\Delta)^{-1}\,. \tag{5.9}$$

The second term in the preceding equation does not contribute to second order because of rotational and parity invariance in the $(u, \bar{u})$ plane. Eq. (5.8) contains traces of powers of a commutator, hence a formal evaluation would hardly lead to any logarithmic divergence. For example the contribution which we are interested in is the trace of a commutator squared:

$$-\frac{1}{2}\,\mathrm{Tr}([(-\Delta)^{-1}, ad(\mu^0)]^2) \tag{5.10}$$

where now the commutator involves only space-time indices. Yet, one term is zero in dimensional regularization, being a tadpole:

$$\mathrm{Tr}((-\Delta)^{-2}, ad(\mu^0)^2) \tag{5.11}$$

while the other term:

$$-\,\mathrm{Tr}((-\Delta)^{-1} ad(\mu^0)(-\Delta)^{-1} ad(\mu^0)) \tag{5.12}$$

furnishes the needed contribution with the correct sign and coefficient to combine with the result from localisation exactly into the perturbative β_0. This seems to be a non-trivial check of our chain of changes of variable.

6. Beta function: the contribution of the rescaling anomaly to the second coefficient

It is an interesting question as to whether Γ reproduces also the second coefficient of the beta function. We do not have a definite answer at the moment. A contribution to β_1, if it exists, can only come from higher order operators, starting with $\mathrm{Tr}(\mu^4)$, that would carry the correct power of g. It is then clear that, to relate $\mathrm{Tr}(\mu^4)$ to $\mathrm{Tr}(\mu^2)$, eq. (2.24) should be employed. A shortcut may perhaps consist in choosing a μ that reproduces to lowest order the gauge propagator as in the two dimensional case. A most natural thing would be to compare our holographic effective action with the Wilsonian effective action of QCD, based on the exact renormalization group [43, 44], if we knew it. In principle an ansatz for the exact Wilsonian effective action should include all higher order operators as it turns out in our holographic case. Yet the simplest ansatz that includes only the lowest order operator $\mathrm{Tr}(F_{\mu\nu}^2)$ was already considered in [45], where it was found, with this ansatz, the following beta function:

$$\frac{\partial g}{\partial \log \Lambda} = -\frac{g^3}{16\pi^2}\frac{11}{3}\left(1 - \frac{g^2}{16\pi^2}\frac{7}{2}\right)^{-1}. \tag{6.1}$$

We would like to give an explanation of this result in the light of our holographic effective action. In the case that the Wilsonian effective action was truncated to the lowest order operator, we should compare the result in [45] with ours without including the higher order operators. In this case our holographic effective action is only and exactly one loop as in the super-symmetric case. At first sight we have a discrepancy, since our result is only one loop while [45] exhibits a NSVZ-like beta function [16]. However, following the analogy with the super-symmetric case, we should remind that in [45] it has been computed the canonically normalised Wilsonian effective action, while we have computed the effective action with the Wilsonian non-canonical normalisation. We can pass from one to the other one by means of a jacobian that takes into account the rescaling anomaly, following [46]. This jacobian has been computed in the $\mathcal{N}=1$ super-symmetric case and in the super-symmetric theory it accounts for the link between the holomorphic beta function, that is only one loop, and the beta function for the canonical coupling. Oddly the analogous computation does not seem to have been ever performed for pure QCD. We fill here this gap. In fact our computation mimics the one in the super-symmetric case, that is therefore recalled below. In the gauge theory with $\mathcal{N}=1$ super-symmetry, without matter multiplets, $\log J$, the logarithm of the jacobian for the rescaling anomaly, receives a contribution from the gluons, V, from the gluinos, $\psi, \bar{\psi}$, form the auxiliary scalar field, D, and from the scalar field that implements the gauge-fixing condition in the Feynman gauge, c:

$$\begin{aligned}\log J = \log g \lim_{M\to\infty}\Bigg(&\mathrm{Tr}_V \exp\Big(-\frac{1}{M^2}(-\Delta_A\delta_{\mu\nu}+i2ad_{F_{\mu\nu}})\Big)- \\ &-\mathrm{Tr}_\psi \exp\Big(-\frac{1}{M^2}\gamma_\mu D_\mu\Big)-\mathrm{Tr}_{\bar{\psi}} \exp\Big(-\frac{1}{M^2}\gamma_\mu D_\mu\Big)+ \\ &+\mathrm{Tr}_D \exp\Big(-\frac{1}{M^2}(-\Delta_A)\Big)-\mathrm{Tr}_c \exp\Big(-\frac{1}{M^2}(-\Delta_A)\Big)\Bigg). \end{aligned} \tag{6.2}$$

Quite obviously in the pure gauge case we get:

$$\log J = \log g \lim_{M\to\infty}\Bigg(\mathrm{Tr}_V \exp\Big(-\frac{1}{M^2}(-\Delta_A\delta_{\mu\nu}+i2ad_{F_{\mu\nu}})\Big)-\mathrm{Tr}_c \exp\Big(-\frac{1}{M^2}(-\Delta_A)\Big)\Bigg). \tag{6.3}$$

We have computed that in this case:

$$\begin{aligned}\log J &= \log g\beta_J\frac{1}{2}\sum_{\mu\neq\nu}\int \mathrm{Tr}(F_{\mu\nu}^2) \\ &= \log g\frac{1}{16\pi^2}\frac{7}{2}\frac{1}{2}\sum_{\mu\neq\nu}\int \mathrm{Tr}(F_{\mu\nu}^2)\,. \end{aligned} \tag{6.4}$$

Now, since the canonical coupling, g_c, and the non-canonical Wilsonian coupling, g_W, are related by the relation:

$$\frac{1}{2g_c^2}=\frac{1}{2g_W^2}-\beta_J\log g_c \tag{6.5}$$

and the Wilsonian coupling is only one loop within our accuracy, we get for the canonical beta function, within our accuracy:

$$\frac{\partial g_c}{\partial\log\Lambda}=-g_c^3\beta_0(1-g_c^2\beta_J)^{-1} \tag{6.6}$$

with:

$$\begin{aligned}\beta_0 &= \frac{1}{16\pi^2}\frac{11}{3} \\ \beta_J &= \frac{1}{16\pi^2}\frac{7}{2} \end{aligned} \tag{6.7}$$

in perfect agreement with the result in [45]. From eq. (6.6)–(6.7) we can read that the contribution of the rescaling anomaly to β_1 in the pure gauge theory is:

$$\beta_{1J} = \beta_0 \beta_J = \frac{1}{(16\pi^2)^2} \frac{77}{6} \tag{6.8}$$

where our convention is that positive β_0, β_1 give origin to a negative beta function. Incidentally this seems to answer a question raised by Shifman [16] about the operator versus canonical conformal anomaly in the pure gauge theory (within our accuracy, since in principle there could be contributions of higher order operators also in the jacobian J). This shows that the missing term that must come from higher order operators to get agreement with the two-loop perturbative result:

$$\beta_1 = \frac{1}{(16\pi^2)^2} \frac{68}{6} \tag{6.9}$$

is:

$$\beta_{1W} = -\frac{1}{(16\pi^2)^2} \frac{9}{6} \, . \tag{6.10}$$

We leave for the future the evaluation of higher order operators in our holographic effective action.

7. Miscellanea

In this section we collect some comments about analogies with the existing literature. In recent years there has been much progress in understanding the non-perturbative physics of super-symmetric gauge theories based on such concepts as effective action, holomorphy, holography and integrable systems. One of the first non-trivial results was the exact NSVZ (canonical) beta function [16] in $\mathcal{N} = 1$ super-symmetric theories, that is related to the fact that the Wilsonian (holomorphic) coupling constant gets only one-loop divergences because of the holomorphic properties of the $\mathcal{N} = 1$ $SUSY$ action written in terms of super-fields. As we have just seen this distinction plays a role in this paper, if we want to relate our Wilsonian holographic effective action to the perturbative canonical one by a computation analogous to the one in [46]. Some time ago a holographic correspondence was found in [10], between a boundary $\mathcal{N} = 4$ four dimensional super-symmetric gauge theory and a bulk five dimensional super-gravity (super-string) theory. New techniques based on integrable systems and holomorphic matrix models have been developed to find the low energy effective action of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ super-symmetric gauge theories in [47] and [48] respectively. The extension of these techniques to non-supersymmetric gauge theories in four dimensions such as QCD_4 encounters considerable difficulties. Though at technical level the construction in this paper of the holographic effective action in large-N QCD_4 for planar self-avoiding loops seems peculiar, at least for the moment, to the $\mathcal{N} = 0$ theory, we would like to interpret the existence of such effective action in the light of some of the concepts that played such an important role in the super-symmetric case. We have already stressed that the construction of the effective action from the loop equation is a case of holography, i.e. of a correspondence between a boundary theory supported on loops, defined by the loop equation, and a bulk theory defined by the effective action. The key point for the existence of this holographic correspondence is holomorphy, that enters here in changing variables in the loop equation in such a way that the quantum contribution to the loop equation in these new variables involves the contour integral of the Cauchy kernel and thus can be reduced to the computation of a regularized residue. In this respect the Cauchy theorem can be considered the oldest and simplest case of holography. Yet, there are two other points of contact with the holography in the sense of [10]. The first one is that to make the quantum residue vanishing is necessary to map the local bulk degrees of freedom of the theory into the boundary cusps. Thus we have a correspondence between the degrees of freedom in the bulk and at the boundary, that is the distinguishing feature of holography in modern sense. In addition this is achieved by means of a conformal mapping that induces a new metric on the fundamental domain in the upper half plane, in analogy with the AdS theory [10]. Notice also that in our approach, in the four dimensional theory, there are logarithmic corrections to such metric due to the conformal anomaly, as already suggested in [35]. It is perhaps remarkable that these features arise directly from first principles, i.e. directly from the loop equation.

The usual unitary monodromy of the connection that enters the loop equation is embedded into a non-unitary family of monodromies whose curvature is either of ASD or of SD type: this non-unitary family plays the analogous role of the holomorphic chiral ring in $\mathcal{N}=1$ super-symmetric gauge theories [49]. In fact the structure of the "classical" holographic action for large-N QCD_4 itself has many analogies with the one of $\mathcal{N}=1$ super-symmetric gauge theories [48]. This effective action is the sum of three terms. The classical action, the Veneziano-Yankielowicz [50] potential (essentially the one-loop contribution) and the contribution of the Konishi anomaly, that occurs as an anomaly in the conformal rescaling of the chiral super-fields in the functional integral. It is interesting to observe that our "classical" holographic action has in fact a very similar structure. The $\mathcal{N}=1$ effective action is defined as a functional of the glueball super-field, that is a scalar composite operator. In our $\mathcal{N}=0$ case the "classical" holographic action is a functional of a composite operator as well, the self-dual and anti-selfdual (in the Z_2 orbifold version) components of the curvature tensor, that play the role of the chiral curvature super-fields W and $\bar{W}$ in the super-symmetric case. There is even a hyper-Kahler structure, often associated to super-symmetry, that is inherited by the moduli fields of the Hitchin systems determined by these ASD or SD fields. Also the "classical" holographic action consists of three terms: the classical action, the localisation determinant (that is only one loop) and the jacobian of the change of variables to the holomorphic gauge. This third term is the analogue of the Konishi anomaly and it is indeed a jacobian under field-dependent gauge transformations living in the complexification of the gauge group. The occurrence of Hitchin systems in the holographic effective action can be considered the analogue of the occurrence of Hitchin systems in the solution for the low energy effective action of the Coulomb branch of $\mathcal{N}=2$ super-symmetric gauge theories [47]. We would like also to recall the reader that the idea of using Hitchin systems to get control over the large-N limit of QCD arises by an energy-entropy argument in [51], that is rooted in the idea of dom-

JHEP06(2005)042

inating the large-N limit by the saddle-point method. Hitchin systems, being completely integrable, admit an "abelianization" map, not to be confused with the one of section 3, that maps parabolic SU(N) Hitchin bundles into U(1) bundles over branched N coverings. Thus the system gets abelianized in such a way that the number of local moduli passes from order of N^2 for the parabolic Hitchin system to order of N for the abelian system on a N covering. Hence for large-N the entropy of the functional integration is re-absorbed into the jacobian of the change of variables of the abelianization map and the usual saddle-point argument for vector-like models applies. In fact we could interpret the holographic map as an attempt to understand in a more precise way, i.e. directly from the loop equation, how Hitchin systems help to get control over the large-N limit of QCD. Finally, the use that we have made of the conformal map to a cuspidal fundamental domain resembles geometric engineering [52] and in particular the fact that only the local behaviour close to the singularity seems to count. The use of cuspidal singularities resembles also the idea [35] of considering five dimensional strings with the four dimensional boundary at a singular point of the metric in the fifth coordinate, as required by the zig-zag symmetry. In turn the zig-zag symmetry plays a crucial role in the vanishing of the quantum residue in the loop equation.

8. Conclusions

We here advocate a drastic change in our way of attempting a solution of the loop equation in the large-N limit of QCD. Rather than trying to solve this equation on the entire loop algebra, a highly non-commutative problem, perhaps of non-hyperfinite nature, we restrict ourselves to the commutative hyper-finite algebra generated by a fixed self-avoiding loop. We look at this problem as a problem in holography, following the analogy with the Cauchy theorem, that reconstructs from the values of a holomorphic function on a loop its values in the internal region. We have constructed a holographic quantum effective action for the eigenvalues of the curvature of a twisted local system, i.e. a centrally extended Hitchin system. Hitchin systems occur in the functional integral by means of the localisation on a microcanonical ensemble labelled by the levels of the ASD (SD) curvature of the gauge connection. We have obtained a holographic form of the loop equation by a change of variable from the connection to the ASD (SD) curvature in the holomorphic gauge. We have written the holographic loop equation for a special connection, whose curvature is of ASD (SD) type. In the new integration variable, i.e. the ASD (SD) curvature in the holomorphic gauge, the quantum contribution to the loop equation is reduced to the computation of a regularized residue, that turns out to be loop independent for self-avoiding loops. To this form of the loop equation we have associated a holographic "classical" effective action. Finally, we have obtained the holographic quantum effective action from the "classical" one by means of a conformal mapping of the loop equation to a cuspidal fundamental domain on which the regularized residue for self-avoiding loops vanishes identically. The conformal map makes possible a peculiar gauge choice and a consequent reduction of the curvature to a Borel sub-algebra. The physically interesting unitary Wilson loops are embedded into the algebra generated by Wilson loops of ASD (SD) type via a limiting procedure that involves a twisting of the Higgs field of the ASD (SD) Hitchin system.

JHEP06(2005)042

In this paper we have computed the first coefficient of the beta function that arises from the holographic "classical" effective action for the connection in the unitary limit, finding exact agreement with the perturbative one-loop result. In addition we have computed that part of the second coefficient which, following the analogy with the rescaling anomaly in $\mathcal{N}=1$ super-symmetric gauge theory, arises from the rescaling in the holographic effective action, in passing from the Wilsonian to the canonical coupling constant. We found also in this case exact agreement with an independent result based on continuum exact renormalization group. The complete second coefficient of the beta function of the holographic effective action is left as a problem for the future.

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JHEP06(2005)042

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JHEP06(2005)042